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A LOWER BOUND ON STRICTLY NON-BLOCKING NETWORKS

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We prove that a strictly non-blocking n-connector of depth k must have $\Omega(n^{1+1/(k-1)})$ edges.

1. Introduction

Let $G=(V, E)$ be a directed acyclic graph with two distinguished subsets of vertices I and J, $I\cap J=\emptyset$, called inputs and outputs respectively. (G, I, J) is said to be *strictly non-blocking* if for any $i\in I$, $j\in J$, and set P of vertex disjoint paths originating in I-$\{i\}$ and terminating in J-$\{j\}$, there is a path from i to j which is vertex disjoint from all paths in P. This definition arises in the study of telephone switching networks. In this model, I and J respectively represent callers and receivers. Our telephone network is designed to allow any number of two-way calls from members of I to members of J, as long as anyone is talking to at most one person at a time. Callers are allowed to hang up and call up a free (i.e. not busy with another caller) receiver. The edges and non-terminal vertices (i.e. $V-I-J$) of G respectively represent telephone wires and switches. A switch selects one incoming and one outgoing wire; the selected wires form paths linking callers to receivers. Distinct paths are not allowed to share switches or wires. If the network is strictly non-blocking then whenever a free caller calls up a free receiver, there will exist a path between them which does not cross any other caller's path.

If $V=V_0\cup V_1\cup\ldots\cup V_k$ with $V_i\cap V_j=\emptyset$ for $i\neq j$, $V_0=I$, $V_k=J$, and if all edges originating in V_i terminate in V_{i+1} for all i, then we call G a graph of depth k. If $|I|=|J|=n$ (here $|A|$ denotes the size of A), then we call G a *strictly non-blocking n-connector*. We shall prove that a strictly non-blocking n-connector of depth k must have $\Omega(n^{1+1/(k-1)})$ edges. This improves the previous known lower bound of $\Omega(n^{1+1/k})$ implied by a result of Pippenger and Valiant (see [2]), and matches the upper bound to within a constant factor for $k=2$ and 3. More generally, the best known upper bound is $O(n^{1+1/m})$ for graphs of depth $2m$ and $2m-1$, for any integer m, due to C. Clos (see [1]).

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2. The lower bound

For $v \in V$ let $D_L(v)$ be those vertices which have an edge originating from them and terminating in v, and $D_R(v)$ be those vertices with an edge terminating in them and originating in v. Let $d_L(v) \equiv |D_L(v)|$ and $d_R(v) \equiv |D_R(v)|$; these are called the left and right degrees of v. First we will prove the result for $k=2$, $V=V_0 \cup V_1 \cup V_2$.

For any $i \in I$, $j \in J$ let $V_{ij} \equiv D_R(i) \cap D_L(j)$.

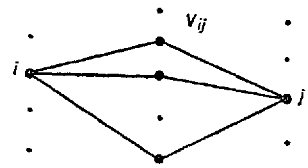

Fig. 1

Lemma 2.1. *For any $i \in I$, $j \in J$ we have*

$$\sum_{v \in V_{ij}} \frac{1}{d_L(v)} + \frac{1}{d_R(v)} \geqq 1. \tag{1}$$

Proof. Let $V_{ij} = \{v_1, \ldots, v_m\}$. Let $v_{a_1}, v_{a_2}, \ldots, v_{a_m}$ be a rearrangement of $v_1, \ldots, v_m$ in increasing d_L order, i.e. $d_L(v_{a_1}) \leqq d_L(v_{a_2}) \leqq \ldots \leqq d_L(v_{a_m})$. Let $v_{b_1}, \ldots, v_{b_m}$ be a rearrangement in increasing d_R order. Assume for the moment that $d_L(v_{a_s}) \geqq s+1$ for all s. Then $d_L(v_{a_1}) \geqq 2$, so there is a vertex $i_{a_1} \in I$ connected to v_{a_1} (i.e. $\in D_L(v_{a_1})$) which is not i. Since $d_L(v_{a_2}) \geqq 3$ there is a vertex $i_{a_2} \in I$ connected to v_{a_2} which is not i or i_{a_1}. Proceeding this way we construct distinct vertices $i_{a_1}, \ldots, i_{a_m}$ none of which are i and such that each i_{a_s} is connected to v_{a_s}. If the same were true for the right degrees, $d_R(v_{b_s}) \geqq s+1$, then we would have each v_{b_s} connected to a distinct j_{b_s} which is not j.

It would then be impossible to find a path from i to j avoiding the m paths $\{i_l \to v_l \to j_l\}_{l=1,\ldots,m}$, since any path from i to j must hit V_{ij}. This contradicts the fact that G is strictly non-blocking.

Thus for some s either $d_L(v_{a_s}) \leqq s$ or $d_R(v_{b_s}) \leqq s$. But then

$$\sum_{v \in V_{ij}} \frac{1}{d_L(v)} + \frac{1}{d_R(v)} \geqq \left(\frac{1}{d_L(v_{a_1})} + \ldots + \frac{1}{d_L(v_{a_s})}\right) + \left(\frac{1}{d_R(v_{b_1})} + \ldots + \frac{1}{d_R(v_{b_s})}\right) \geqq$$

$$\geqq s \times \frac{1}{d_L(v_{a_s})} + s \times \frac{1}{d_R(v_{b_s})} \geqq 1. \quad \blacksquare$$

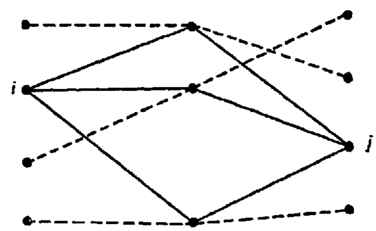

Fig. 2

Theorem 2.2. *Let G be a strictly non-blocking n-connector of depth 2. Then $|E| \geqq n^2$.*

Proof. Summing equation (1) over all i and j we have

$$n^2 \leqq \sum_{i\in I, j\in J} \sum_{v\in V_{ij}} \frac{1}{d_L(v)} + \frac{1}{d_R(v)} =$$

$$= \sum_{v\in V_1} \left(\frac{1}{d_L(v)} + \frac{1}{d_R(v)}\right) (\#\text{ of } V_{ij}\text{'s containing } v) =$$

$$= \sum_{v\in V_1} \left(\frac{1}{d_L(v)} + \frac{1}{d_R(v)}\right) d_L(v) d_R(v) = \sum_{v\in V_1} (d_R(v) + d_L(v)) = |E|. \quad \blacksquare$$

Now we prove the depth k case, $V = V_0 \cup V_1 \cup \ldots \cup V_k$. For $i \in I$ and $j \in J$ let A_{ij} be those vertices of V_1 which lie on a path from i to j, and B_{ij} those of V_{k-1} on paths from i to j.

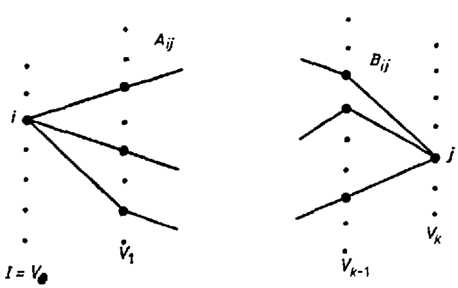

Fig. 3

Lemma 2.3. *For any $i \in I, j \in J$ we have*

$$\sum_{v\in A_{ij}} \frac{1}{d_L(v)} + \sum_{v\in B_{ij}} \frac{1}{d_R(v)} \geqq 1. \tag{2}$$

Proof. If not, then there are distinct vertices $i_1, \ldots, i_m$ of I, none of which is i, each connected to a distinct member of A_{ij}, and vertices $j_1, \ldots, j_{m'}$ of J similarly connected to B_{ij}. Now connect as many members of A_{ij} as possible to members of B_{ij} by vertex disjoint paths, and complete the paths from $i_1, \ldots, i_m$ to $j_1, \ldots, j_{m'}$. Then there is no path from A_{ij} to B_{ij} avoiding this set of paths, and thus no path from i to j avoiding this set. This contradicts the strictly non-blocking property of G. $\blacksquare$

Lemma 2.4. *Let G be a strictly non-blocking n-connector of depth k for which*

$$d_L(v) \leqq \frac{8|E|}{n}, \quad d_R(v) \leqq \frac{8|E|}{n}, \quad \forall v \in V.$$

Then

$$|E| \geqq \left(\frac{1}{8}\right)\left(\frac{1}{2}\right)^{1/(k-1)} n^{k/(k-1)}.$$

Proof. For $j \in J$, let A_j denote the set of vertices in V_1 which have a path to j; let B_i, for $i \in I$, denote those of V_{k-1} which lie on some path from i. Since $d_R(v) \leqq$

$\leqq 8|E|/n$ for all v, we have that for any $j\in J$,

$$|A_j| \leqq \left(\frac{8|E|}{n}\right)^{k-1}.$$

Similarly,

$$|B_i| \leqq \left(\frac{8|E|}{n}\right)^{k-1}.$$

Now summing equation (2) over all i, j yields

$$n^2 \leqq \sum_{ij} \sum_{v\in A_{ij}} \frac{1}{d_L(v)} + \sum_{ij} \sum_{v\in B_{ij}} \frac{1}{d_R(v)}$$

$$\sum_j \sum_{v\in A_j} (\#\, i\text{'s connected to } v) \frac{1}{d_L(v)} + \sum_i \sum_{v\in B_i} (\#\, j\text{'s connected to } v) \frac{1}{d_R(v)} =$$

$$= \sum_j \sum_{v\in A_j} 1 + \sum_i \sum_{v\in B_i} 1 \leqq \sum_j \left(\frac{8|E|}{n}\right)^{k-1} + \sum_i \left(\frac{8|E|}{n}\right)^{k-1} = 2n \left(\frac{8|E|}{n}\right)^{k-1}$$

(since $|J|+|I|=2n$). Thus

$$|E|^{k-1} \geqq \left(\frac{1}{8}\right)^{k-1} \left(\frac{1}{2}\right) n^k$$

and the lemma follows. ▮

Theorem 2.5. *Let G be a strictly non-blocking n-connector of depth $k\geqq 3$. Then*

$$|E| \geqq \frac{1}{32} n^{1+1/(k-1)}.$$

Proof. Let V' be the set of vertices with d_L of d_R exceeding $4|E|/n$. Since $\sum_{v\in V} d_L(v) = |E|$, there are at most $n/4$ vertices with d_L more than $4|E|/n$. Similarly there are at most $n/4$ vertices with d_R more than $4|E|/n$. Thus $|V'|\leqq n/2$. Let P be a maximal set of vertex disjoint paths from I to J such that each path hits at least one member of V'. Then the subgraph induced on the vertex set $\tilde{V}=V-V'-\{\text{vertices in } P\}$ is strictly non-blocking. Furthermore $\tilde{V}$ has (# inputs)=(# outputs)$\geqq n/2$. Applying Lemma 2.4 to $\tilde{V}$ yields that the number of edges in the induced subgraph is at least

$$\left(\frac{1}{8}\right)\left(\frac{1}{2}\right)^{1/(k-1)} \left(\frac{1}{2}n\right)^{k/(k-1)} = \left(\frac{1}{8}\right)\left(\frac{1}{2}\right)^{1+2/(k-1)} n^{1+1/(k-1)} \geqq \frac{1}{32} n^{1+1/(k-1)}$$

(remember $k\geqq 3$ so $2/(k-1)\leqq 1$). The theorem follows.

References

[1] C. Clos, A Study of non-blocking switching networks, *Bell. Sys. Tech. J.*, **32** (1953), 406—424.
[2] N. Pippenger and L. Valiant, Shifting graphs and their applications, *J. ACM*, **23** (1976), 423—432.

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